J. Fluid Mech. (1978), *vol.* 85, *part* 3, *pp.* 479–496
Printed in Great Britain

Flow between a stationary and a rotating disk with suction

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(Received 4 March 1977 and in revised form 8 August 1977)

The equations for the viscous flow between two coaxial infinite disks, one stationary and the other rotating, are solved numerically. The effects of applying a uniform suction through the rotating disk are determined. Initially, the fluid and disks are at rest. The angular velocity of one disk and the amount of suction through it are gradually increased to specific values and then held constant. At large Reynolds numbers R, the equilibrium flow approaches an asymptotic state in which thin boundary layers exist near both disks and an interior core rotates with nearly constant angular velocity. We present graphs of the equilibrium flow functions for $R = 10^4$ and various values of the suction parameter a ($0 \leqslant a \leqslant 2$). When $a = 0$, the core rotation rate ω_c is 0·3131 times that of the disk. Fluid near the rotating disk is thrown centrifugally outwards. As a increases, ω_c increases and the centrifugal outflow decreases. When $a > 1{\cdot}3494$, the core rotation rate exceeds that of the disk and the radial flow near the rotating disk is directed inwards. We also present accurate tabular results for two flows of special interest: (i) the flow between a stationary and a rotating disk with no suction ($a = 0$) and (ii) Bödewadt flow. The latter can be obtained by suitable scaling of the boundary-layer solution near the stationary disk for any $a \geqslant 0$. Since several solutions to the steady-state equations of motion have been reported, the question arises as to whether other solutions to the time-dependent equations of motion with zero initial conditions can be found. We exhibit a rotational start-up scheme which leads to an equilibrium solution in which the interior fluid rotates in a direction opposite to that of the disk.

1. Introduction

The study of the flow of a viscous incompressible fluid between two infinite rotating disks is of both theoretical and practical importance. The problem's origins lie in the investigation of flow above a single rotating disk. Von Kármán (1921), by assuming the axial velocity to be independent of the radius, reduced the Navier–Stokes equations for the steady flow due to a rotating disk to a set of ordinary differential equations; these have been solved numerically by Cochran (1934) and by many others. Bödewadt (1940) obtained a numerical description of the flow over a stationary plane when the fluid infinitely far away from it is in a state of solid-body rotation. Batchelor (1951) extended the discussion to a family of rotationally symmetric flows whose members are distinguished by the ratio of the angular velocity of the fluid at infinity to that of the disk. For non-negative values of this ratio, Rogers & Lance (1960) solved the flow equations numerically. The effects of suction through the rotating disk were considered by Stuart (1954), Rogers & Lance (1960), Evans (1969), Ockendon (1972) and Bodonyi (1975); Kuiken (1971) investigated the effects on the flow of blowing through the disk.

An extension of von Kármán's analysis will also reduce to ordinary differential equations the Navier–Stokes equations for steady rotationally symmetric flows between two infinite coaxial disks, either or both of which may have a uniform suction through its surface (Batchelor 1951). Time-varying flow can also be considered, both for the two-disk problem (Pearson 1965; H. P. Greenspan 1968) and for the single-disk problem (Benton 1966). The two-disk problem has been studied extensively both theoretically and numerically, particularly in the steady-state case with no suction through the disks. The nature of the steady flow when the disks are counter-rotating, especially when they are rotating with the same speed but in the opposite sense, has provoked a number of investigations, not all of which are in agreement. Among these are studies by Batchelor (1951), Stewartson (1953), Lance & Rogers (1962), Pearson (1965), Tam (1969), D. Greenspan (1972), McLeod & Parter (1974) and Matkowsky & Siegmann (1976). Less attention has been paid to the case of disks co-rotating with non-zero angular velocities, possibly because early conjectures (Batchelor 1951; Stewartson 1953) about the character of this flow were in agreement. Some computations have been made by Lance & Rogers (1962).

In this paper we discuss the remaining case, in which one disk is at rest and the other rotates. We are particularly interested in the large Reynolds number solution when there is a uniform suction through the rotating disk. The solution for this case when no suction is present has received much attention in the literature. Lance & Rogers (1962) solved the steady-state equations for Reynolds numbers up to $R = 441$. Pearson (1965), for Reynolds numbers as large as $R = 10^3$, studied the time evolution of the flow from an impulsive start with zero initial conditions. Steady-state calculations were made by Benton (1968) up to $R = 2500$ and by Cooper & Reshotko (1975) for $R = 2852$. These computations all indicate that the fluid, except for that in thin layers near each disk, rotates at a uniform angular velocity intermediate between those of the two disks.

It appears that the equations may have other solutions as well. Mellor, Chapple & Stokes (1968) and Roberts & Shipman (1976) presented numerical evidence that for a single Reynolds number many steady-state solutions are possible. The validity of some of these, known as multiple-cell solutions, has been questioned (Parter 1977, private communication). Nevertheless, more than one single-cell solution seems to exist. Watts (1974) made an asymptotic analysis of various such solutions.

From a theoretical standpoint, the question of the existence of solutions to two-disk flow problems has not yet been settled, let alone that of uniqueness. Progress concerning the existence of solutions has been made by Hastings (1970), McLeod & Parter (1974) and Elcrat (1975). McLeod & Parter (1974, 1977) presented some rigorous discussions about the behaviour of solutions. For the related problem in which both disks are at rest and there is suction or blowing through the disks, Elcrat (1976) has demonstrated both existence and uniqueness.

In our work, which is primarily concerned with the effect on the flow of a uniform suction through the rotating disk, we consider the full time-dependent problem rather than the steady-state equations discussed by most authors. The fluid and disks are initially at rest; the angular velocity of one disk and the amount of suction through it are gradually increased to specified values and then held constant. In a sense this complements the impulsive-start problem of Pearson (1965). We shall present results only for the flow which exists after the disk velocity and the suction have been held constant for a long time. We call this the equilibrium flow, to distinguish it from the solution obtained from the steady-state equations of motion. It is possible also to consider the time evolution of the flow, but we shall not do so here. Our numerical scheme is a general one for solving a coupled system of nonlinear second-order partial differential equations with one space variable and one time variable. It involves Galerkin's method in space using B-splines, and an extrapolated backwards Euler scheme in time.

We obtained solutions for Reynolds numbers between $R = 10^2$ and $R = 10^4$. Since our computations and those of others leave little doubt about the asymptotic nature of the solution for large Reynolds numbers, we present results only for $R = 10^4$. For the rotational start-up scheme described, we find that thin boundary layers exist at both disks and are separated by a core region which rotates at a nearly uniform non-zero angular velocity ω_c. As the suction through the rotating disk increases, the boundary layers become thinner while the interior rotation rate increases and can even exceed that of the disk.

Since our numerical scheme permits automatic computation of the equilibrium solution to a specified accuracy, we also present accurate results in tabular form for two flows of interest: (i) that between a stationary and a rotating disk with no suction and (ii) Bödewadt flow (obtained from the boundary-layer solution near the stationary disk).

The question as to whether other solutions to the steady-state equations of motion can actually be obtained from the full time-dependent equations of motion with zero initial conditions is discussed briefly. A rotational start-up scheme is exhibited which leads to an equilibrium solution in which the interior fluid rotates in a direction opposite to that of the disk.

2. Equations of motion

We study the viscous flow between two parallel infinite disks. Initially, the fluid and the disks are at rest. At time $t = 0$, one disk is set in motion, its angular velocity being gradually increased to a value Ω and then held constant. Suction is also applied to the rotating disk at $t = 0$ and is gradually increased to a specified level, after which it is held constant. The equilibrium solution to this time-dependent problem may be obtained by considering the flow after the angular velocity and suction have been held constant for a long time.

For axisymmetric flow of an incompressible fluid, the continuity equation and the Navier–Stokes equations are written in dimensionless form in cylindrical co-ordinates as

$$(ru)_r + (rw_z) = 0, \tag{2.1}$$

$$u_t + u_r u - v^2/r + u_z w = -\rho^{-1}p_r + R^{-1}(u_{rr} + u_r/r + u_{zz} - u/r^2), \tag{2.2}$$

$$v_t + v_r u + uv/r + v_z w = R^{-1}(v_{rr} + v_r/r + v_{zz} - v/r^2), \tag{2.3}$$

$$w_t + w_r u + w_z w = -\rho^{-1}p_z + R^{-1}(w_{rr} + w_r/r + w_{zz}). \tag{2.4}$$

Here $R = d^2\Omega/\nu$ is the Reynolds number, where d is the disk spacing and ν is the kinematic viscosity. Distance, time, velocity and pressure divided by density have been normalized with respect to d, Ω^{-1}, Ωd and $(\Omega d)^2$ respectively. The subscripts denote partial derivatives.

If the axial velocity w is assumed to have the form $w = R^{-\frac{1}{2}}H(t,z)$, then the following system of equations must be satisfied:

$$\phi_z = 0, \quad 2F + R^{-\frac{1}{2}}H_z = 0, \tag{2.5), (2.6}$$

$$F_t = G^2 - F^2 - R^{-\frac{1}{2}}F_z H - \phi + R^{-1}F_{zz}, \tag{2.7}$$

$$G_t = -2FG - R^{-\frac{1}{2}}G_z H + R^{-1}G_{zz}, \tag{2.8}$$

$$H_t = -R^{-\frac{1}{2}}H_z H - R^{-\frac{1}{2}}\Phi_z + R^{-1}H_{zz}, \tag{2.9}$$

where

$$u = rF(t,z), \quad v = rG(t,z), \quad w = R^{-\frac{1}{2}}H(t,z), \tag{2.10)–(2.12}$$

$$\rho^{-1}p = \tfrac{1}{2}r^2\phi(t) + R^{-1}\Phi(t,z). \tag{2.13}$$

After the velocity functions F, G and H and the radial pressure function ϕ have been determined from the sixth-order coupled system (2.5)–(2.8), the axial pressure function Φ can be obtained from (2.9). In fact, in the equilibrium limit, Φ can be written explicitly in terms of F and H. We shall thus concern ourselves only with the solution of (2.5)–(2.8).

If the rotating disk is located at $z = 0$ and the stationary disk is at $z = 1$ in the normalized co-ordinate system, then the boundary conditions corresponding to no-slip flow with uniform suction through the rotating disk are

$$\left.\begin{aligned} F(t,0) = 0, \quad G(t,0) = \omega(t), \quad H(t,0) = -a(t), \\ F(t,1) = G(t,1) = H(t,1) = 0. \end{aligned}\right\} \tag{2.14}$$

The angular-velocity function $\omega(t)$ and the suction function $a(t) \geqslant 0$ are prescribed. The normalization is such that $\omega(t)$ will have a value of unity at large values of t and $a(t)$ will tend to a value a, termed the suction parameter. We chose functions $\omega(t)$ and $a(t)$ which increase very slowly, rising linearly from zero at $t = 0$ to their final values at $t = R^4$. After that time, all boundary conditions are constant.

Initially, the fluid is at rest and we have

$$F(0,z) = G(0,z) = H(0,z) = \phi(0) = 0. \tag{2.15}$$

In order to obtain an equilibrium solution, we computed out to the very generous time $t = 100R^4$, then refined the spatial mesh in the manner described in the next section and computed out to the final time $t_f = 200R^4$. The program allowed large time steps to be taken when the solution changed little with time and, indeed, very large time steps were taken after the boundary conditions had been held constant for a while.

3. Method of numerical solution

We used a FORTRAN software package POST (Partial and Ordinary differential equation solver in Space and Time) which is designed to solve a coupled system of nonlinear second-order partial differential equations which are a function of one space and one time variable, and associated ordinary differential equations which are a function of time. The POST package is built upon the PORT library of portable FORTRAN subprograms for numerical mathematics (Fox, Hall & Schryer 1976) and will ultimately be an offering in that library.

The numerical method used in the POST routine is described elsewhere (Schryer 1976). In brief, the Rayleigh–Ritz–Galerkin method is used to project each component of the solution vector $\mathbf{u}(t,z)$ at a given instant of time onto the space spanned by B-splines. (These are piecewise polynomials of a specified degree which are defined on a specified spatial mesh and which satisfy certain continuity conditions at the mesh points.) A computationally convenient basis for such a space exists. This method reduces the partial differential equations in space and time to ordinary differential equations in time for the coefficients $U_{ji}(t)$ in the expansion

$$u_i(t,z) = \sum_j U_{ji}(t)\, B_j(z), \tag{3.1}$$

where the $B_j(z)$ are the B-spline basis functions. These ordinary differential equations are then solved by a linearized backwards Euler method.

The routine is capable of *automatically* computing the solution to a specified accuracy. The user specifies how accurately the solution in time is to be computed and the routine automatically determines the sizes of the time steps needed to maintain that accuracy. We also modified the routine to enable the user to specify an error tolerance for the equilibrium solution. The modified routine automatically refines the spatial mesh to obtain a solution with that accuracy.

For our problem, we specified that the solution should be approximated by cubic B-splines, i.e. by piecewise polynomials of degree less than or equal to three. We usually computed each function to an accuracy of 0·1% relative to its maximum absolute value.

One of the elegant features of the POST routine is its ability to use a non-uniform spatial mesh for the B-splines. This is particularly important in our problem since at large Reynolds numbers the solution changes rapidly in thin boundary layers near the disks. In the interior, however, the solution is nearly constant. If the flow functions were to be approximated accurately by functions defined on a uniform mesh, the mesh would have to be quite fine and computations would be relatively expensive. However, if a non-uniform mesh is used, it can be made finer where the solution varies rapidly and coarser where it varies slowly.

The construction of the spatial mesh and the procedure for refining it automatically are peculiar to this specific problem and are not included in the description of the general-purpose POST routine (Schryer 1976). We feel that it is worthwhile to discuss these matters in some detail here.

After some experimentation, we discovered that the widths of the boundary layers near the rotating and stationary disks could be parameterized as functions of the Reynolds number R and the suction parameter a. The boundary layer on the stationary disk is approximately four times as thick as that on the rotating disk, in agreement with results of Reshotko & Rosenthal (1971). For sufficiently large R, the thickness varies as $R^{-\frac{1}{2}}$, as many investigators have shown. As will be discussed in § 4 [see equations (4.1) and (4.2)], we found empirically that the thickness also varies as $(2\frac{1}{3})^{-a}$. We specified the thickness of the boundary layer on the rotating disk to be

$$\delta = 5R^{-\frac{1}{2}}(2\tfrac{1}{3})^{-a}. \tag{3.2}$$

The B-spline mesh was parameterized to have M equally spaced points in each of the intervals $[0, \delta]$ and $[\delta, 1-4\delta]$ and $4M$ points in the interval $[1-4\delta, 1]$. There is a conjecture that the numerical stability (condition) of the B-spline basis degenerates as e^{ρ}, where ρ is the ratio of the largest and the smallest mesh spacing. Our experience is consistent with this conjecture. On a Honeywell HIS-6000 computer, in double precision (18 decimal digits) the time step once went to zero in the solution process. We then realized that in that instance $e^{\rho} \approx 10^{13}$ and there were insufficient digits in the computer solution. Thus for large R we altered the mesh specification and used enough points in the middle interval $[\delta, 1-4\delta]$ to keep $\rho \leqslant 10$. These extra points are not needed for accuracy in that region, but rather to keep the scheme numerically stable.

With this parameterization of the B-spline mesh, we still needed to determine what value of M to use to obtain an equilibrium solution with a specified accuracy. We accomplished this *automatically* in the following manner. We first solved the equations using a small value of M, eventually obtaining an approximation to the equilibrium solution. We then refined the mesh by increasing M to approximately $\frac{4}{3}$ the first value, used the former approximate equilibrium solution as an initial condition and, with very little additional computation, obtained a better approximate equilibrium solution.

We then used a PORT (Fox *et al.* 1976) library facility (EEBSF) to estimate the error in a B-spline approximation to a function. Given two different numerical solutions, on different B-spline meshes (one of which is finer than the other), the subprogram EEBSF can be used to estimate the error in each approximate solution. For reasons which we shall not discuss here, the error estimate for the solution determined on the coarser mesh is extremely reliable, but that determined on the finer mesh is rather unreliable. Thus we base all computations on the error estimate for the coarse solution. With the aid of the subprogram EEBSF, we obtain a reliable estimate of the error ϵ_c in the coarse equilibrium solution.

Now the rate of convergence of the solution in space is $O(h^k)$, where h is the mesh spacing and $k = 4$ is the order of the B-spline (de Boor 1968). We may write

$$\epsilon_c = \|\mathbf{u}-\mathbf{u}_c\| = Ch_c^k, \tag{3.3}$$

where $\mathbf{u}_c$ is the coarse solution, C is an unknown constant and h_c is the maximum coarse mesh spacing. We wish to determine the 'optimal' mesh spacing needed for an error ϵ in the equilibrium solution. Since

$$\epsilon = \|\mathbf{u}-\mathbf{u}_0\| = Ch_0^k, \tag{3.4}$$

we obtain from (3.3) and (3.4)

$$\epsilon_c/\epsilon = (h_c/h_0)^k = (M_0/M_c)^k \tag{3.5}$$

and

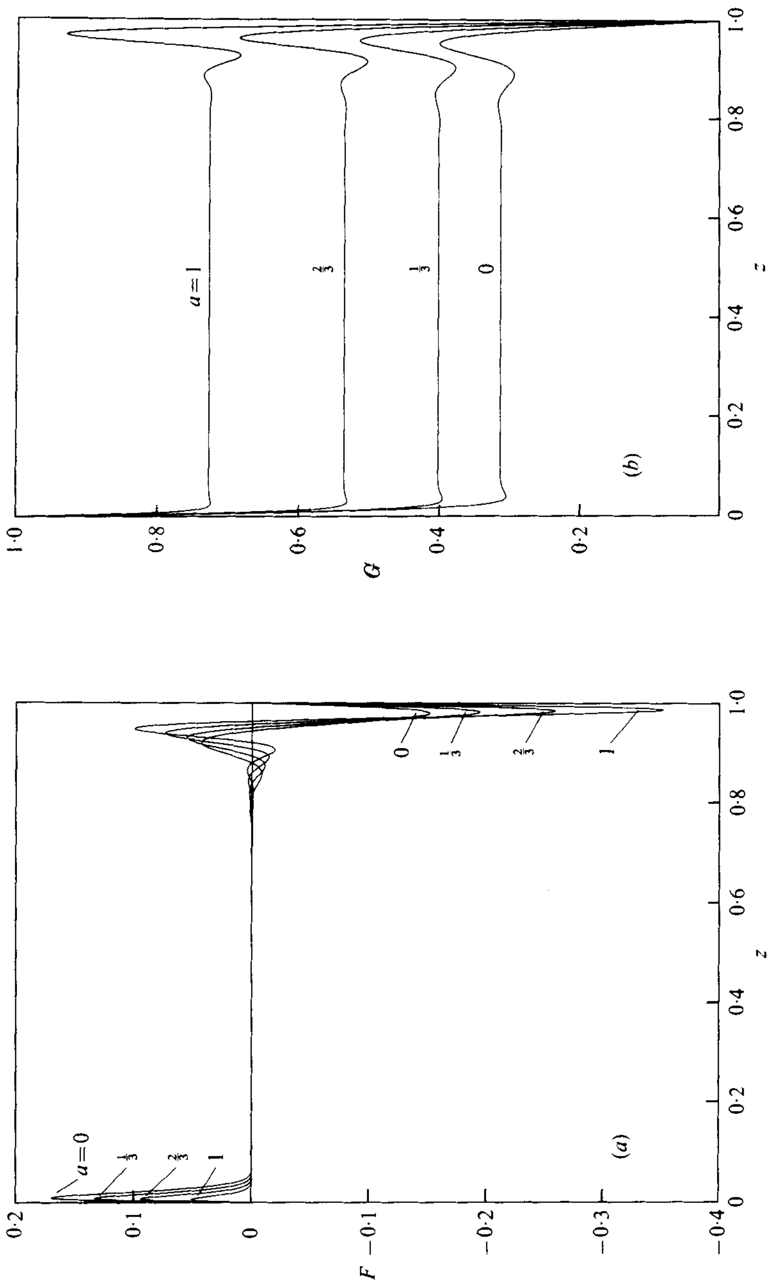
$$M_0 = M_c(\epsilon_c/\epsilon)^{1/k},$$

where M_c is the value of M for the coarse mesh and M_0 is the value for the 'optimal' mesh. Thus, with two approximate solutions in hand, we may estimate the 'optimal' number of mesh points to be used in approximating the solution in space. We then solve (2.5)–(2.8) with $M = M_0$, using the previous approximate equilibrium solution as an initial condition. In order to get an accurate error estimate for *this* solution, we

z	F	G	H
0·000	0·0000	1·0000	0·0000
0·004	0·1275	0·7982	−0·0586
0·008	0·1680	0·6310	−0·1810
0·012	0·1628	0·5072	−0·3152
0·016	0·1377	0·4225	−0·4362
0·020	0·1073	0·3683	−0·5342
0·024	0·0789	0·3358	−0·6085
0·028	0·0553	0·3179	−0·6618
0·032	0·0372	0·3090	−0·6985
0·036	0·0239	0·3056	−0·7226
0·040	0·0148	0·3050	−0·7378
0·044	0·0086	0·3059	−0·7470
0·048	0·0047	0·3072	−0·7522
0·052	0·0023	0·3086	−0·7549
0·056	0·0009	0·3098	−0·7562
0·060	0·0002	0·3108	−0·7566
0·064	−0·0002	0·3116	−0·7565
0·068	−0·0003	0·3122	−0·7563
0·072	−0·0003	0·3125	−0·7561
0·076	−0·0003	0·3128	−0·7558
0·080	−0·0002	0·3129	−0·7556
0·084	−0·0002	0·3130	−0·7554
0·088	−0·0001	0·3131	−0·7553
0·092	−0·0001	0·3131	−0·7552
0·096	−0·0001	0·3131	−0·7552
0·100	0·0000	0·3131	−0·7551
0·150	0·0000	0·3131	−0·7551
0·200	0·0000	0·3131	−0·7551
0·250	0·0000	0·3131	−0·7551
0·300	0·0000	0·3131	−0·7551
0·350	0·0000	0·3131	−0·7551
0·400	0·0000	0·3131	−0·7551
0·450	0·0000	0·3131	−0·7551
0·500	0·0000	0·3131	−0·7551
0·550	0·0000	0·3131	−0·7551
0·600	0·0000	0·3131	−0·7550
0·650	0·0001	0·3131	−0·7551
0·700	−0·0001	0·3133	−0·7555
0·750	−0·0001	0·3124	−0·7528
0·800	0·0017	0·3147	−0·7629
0·850	−0·0079	0·3120	−0·7392
0·900	0·0252	0·2993	−0·7538
0·950	−0·0279	0·4007	−1·0209
1·000	0·0000	0·0000	0·0000

TABLE 1. The zero-suction solution: $R = 10^4$ and $a = 0$. The values of the derivatives F_z and G_z at $z = 0$ and 1 are given in the text.

refine the mesh by a factor of approximately $\frac{4}{3}$ and find a solution once more. If the error is still not small enough, we repeat the process. This scheme has worked efficiently and well for all data sets studied. When $R = 10^4$, M_0 ranged from $M_0 = 12$ for $a = 1$ to $M_0 = 27$ for $a = 0$ when the error was specified to be $0{\cdot}1\%$.



FIGURES 1(*a*) and (*b*). For legend see facing page.

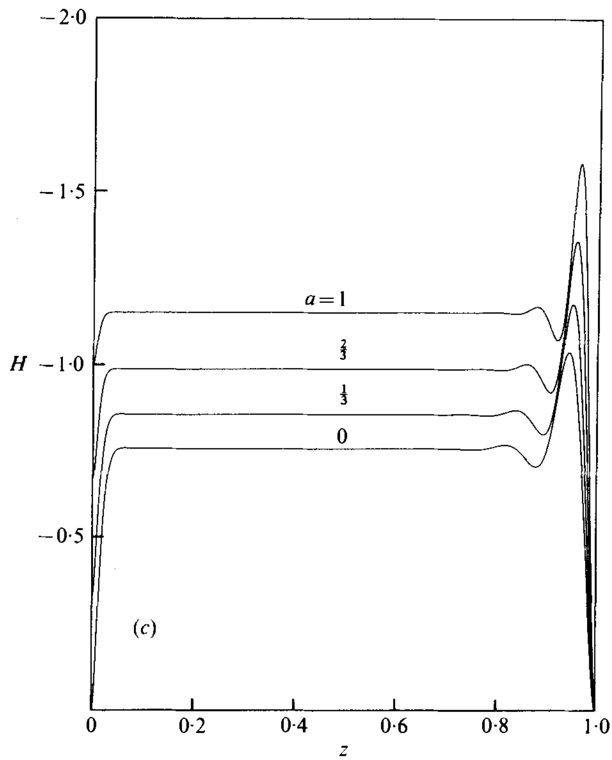
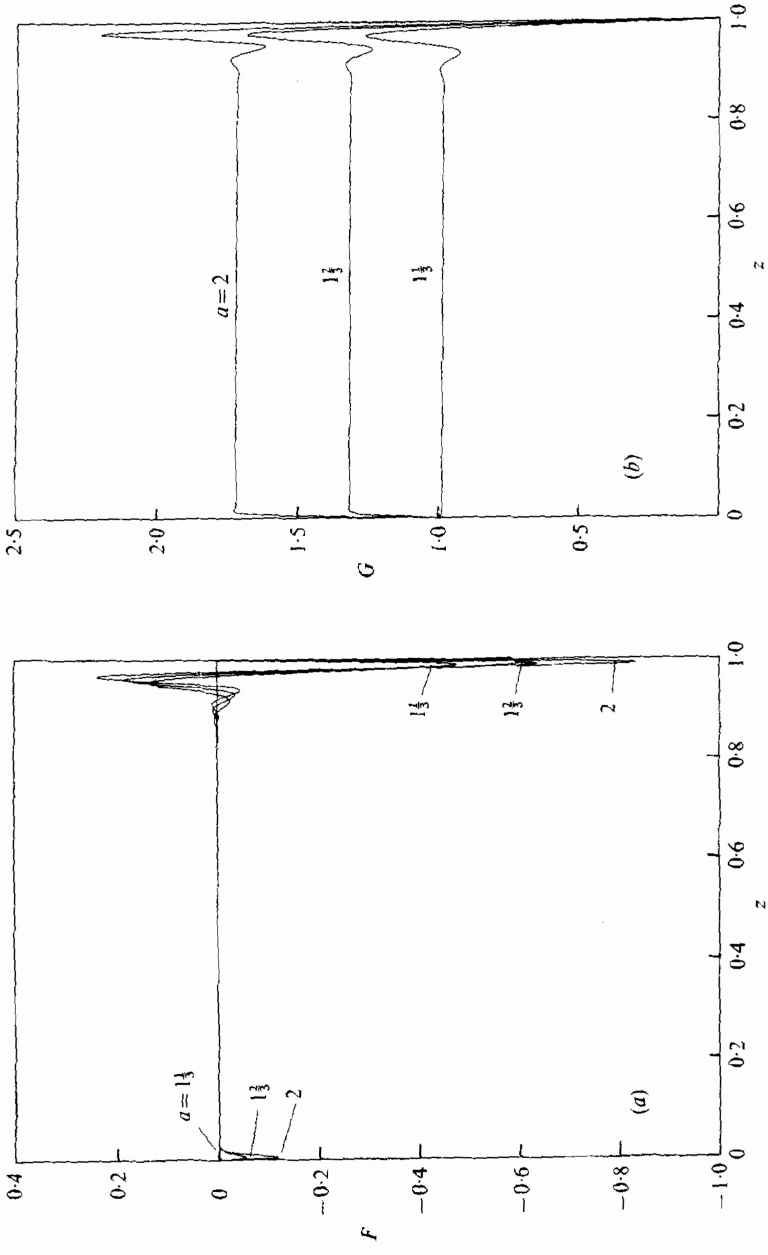


FIGURE 1. The equilibrium profiles of (*a*) F, (*b*) G and (*c*) H; $R = 10^4$; $a = 0, \frac{1}{3}, \frac{2}{3}, 1$.

4. Results

We performed computations with Reynolds numbers $R = 10^2$, 10^3 and 10^4 and with various values of the suction parameter a in the range $0 \leqslant a \leqslant 2$. For all such values of a, our conclusions about the equilibrium flow dependence on R are in complete agreement with the results discussed in some detail by Benton (1968) and by Reshotko & Rosenthal (1971) for the case $a = 0$. Accordingly, we present graphical results here only for the fully asymptotic case $R = 10^4$. After comparing our results for the case $a = 0$ with those of others, we shall concern ourselves primarily with the dependence of the equilibrium flow on the suction parameter.

Figures 1 (*a*), (*b*) and (*c*) display the equilibrium profiles of the radial, angular and axial functions for $a = 0, \frac{1}{3}, \frac{2}{3}$ and 1 with $R = 10^4$. Each function was computed to an accuracy of 0·1 % relative to its maximum absolute value. When there is no suction ($a = 0$), well-developed boundary layers exist near both disks and are separated by a core region in which the motion is very nearly that of a rigid body. Near the rotating disk, the fluid is drawn towards the disk (figure 1 *c*) and is thrown centrifugally outwards (figure 1 *a*). In the core region, the fluid swirls towards the rotating disk (figures 1 *b*, *c*). The interior rotation rate is 0·3131 that of the disk (figure 1 *b*). The flow exhibits an oscillatory nature in the boundary layer near the stationary disk. Regions of inward and outward swirling flow are present; the predominant flow, which is closest to the stationary disk, is one of inward swirling. Some of the fluid rotates faster



FIGURES 2(*a*) and (*b*). For legend see facing page.

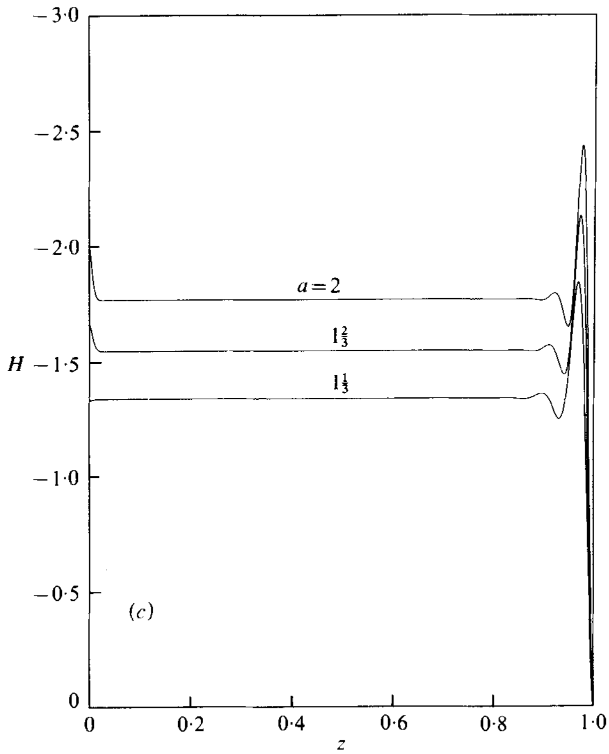


FIGURE 2. The equilibrium profiles of (a) F, (b) G and (c) H; $R = 10^4$; $a = 1\frac{1}{3}, 1\frac{2}{3}, 2$. Note that the scales differ from those in figure 1.

than that in the core region. Axially (figure 1*c*), the fluid moves rapidly away from the stationary disk, then slows down somewhat as it enters the core region. Our computations indicate, incidentally, that an oscillatory behaviour is also present to a lesser extent near the rotating disk.

Pearson's (1965) equilibrium solutions, which were computed for $R = 10^2$ and 10^3 with no suction through the rotating disk, and our (unillustrated) results for $R = 10^2$ and 10^3 and $a = 0$ are in agreement except for slight numerical differences. The steady-state solutions of Lance & Rogers (1962), obtained for Reynolds numbers up to $R = 441$, those of Benton (1968) for $R = 2500$ and those of Cooper & Reshotko (1975) for $R = 2852$ are also consistent with our equilibrium results ($a = 0$).

Although our equilibrium solutions for $R = 10^2$ and $a = 0$ and the steady-state computations of D. Greenspan (1972) are in good agreement, our results for $R = 10^3$ are distinctly different. However, the fact that Greenspan's computer results are incorrect for the case of disks rotating with the same speed but in the opposite sense has been established analytically (McLeod & Parter 1974; Schultz & Greenspan 1974). There is also evidence (Nguyen, Ribault & Florent 1975) that his results are incorrect for the problem at hand. We suspect that the inaccuracy for large R is due to his having used 50 uniformly spaced mesh points in the integration interval, independent of R, even though the boundary layers are very thin at large values of R.

We are, we believe, the first to carry out numerical computations to $R = 10^4$. In table 1, we present accurate results for the case $R = 10^4$ and $a = 0$. Our claim that

a	ϕ
0	0·0980
$\frac{1}{3}$	0·1614
$\frac{2}{3}$	0·2866
1	0·5288
$1\frac{1}{3}$	0·9717
$1\frac{2}{3}$	1·7327
2	2·9605

TABLE 2. The pressure function ϕ for various values of the suction parameter a.

each function is accurate to 0·1% relative to its maximum absolute value is probably quite conservative. We also find that $F_z(0) = 47{\cdot}3$, $G_z(0) = -52{\cdot}4$, $F_z(1) = 16{\cdot}5$ and $G_z(1) = -13{\cdot}5$. Solutions for these functions at other (sufficiently large) values of R may be obtained by scaling the thicknesses of the boundary layers and the solutions within them by $R^{-\frac{1}{2}}$. The magnitudes of F, G and H remain the same. We do not tabulate the solution near the stationary disk at very many values of z because it can also be obtained by suitable scaling of the Bödewadt solution, which appears later in table 3.

Returning to figures 1 (a), (b) and (c), we observe that increasing the suction causes the fluid in the interior to rotate more rapidly and to flow more rapidly towards the rotating disk. In fact, if a is large enough, the rotation rate ω_c of the core is larger than that of the rotating disk, as is illustrated in figures 2 (a), (b) and (c).

Both boundary layers become thinner as a increases. Near the stationary disk, the oscillatory nature of the flow becomes more pronounced. The flow description in the boundary layer near the rotating disk differs according to whether the interior rotation rate is smaller or larger than that of the disk. If it is smaller, then near the disk the fluid is thrown centrifugally *outwards*. As the amount of suction increases, the radial outflow decreases. When $a = 1{\cdot}3494$, the interior rotation rate matches that of the disk and the boundary layer vanishes. As the amount of suction and the interior rotation rate increase further, the boundary layer reappears, but the radial flow is an *inward* flow.

The numerical results for the equilibrium value of ϕ, which appears in the pressure function (2.13), are given in table 2 ($R = 10^4$, a varying). It has been suggested heuristically (Reshotko & Rosenthal 1971) and has recently been shown rigorously (Parter 1977, private communication) that, in the limit of large R, $\phi^{\frac{1}{2}}$ tends to ω_c, the angular velocity of the core. To the accuracy of our computations, we have equality: $\omega_c = \phi^{\frac{1}{2}}$ when $R = 10^4$ for all values of a considered.

Incidentally, the formula

$$\omega_c(a) = \omega_c(0)\,(2\tfrac{1}{3})^a, \quad \omega_c(0) = 0{\cdot}3131, \tag{4.1}$$

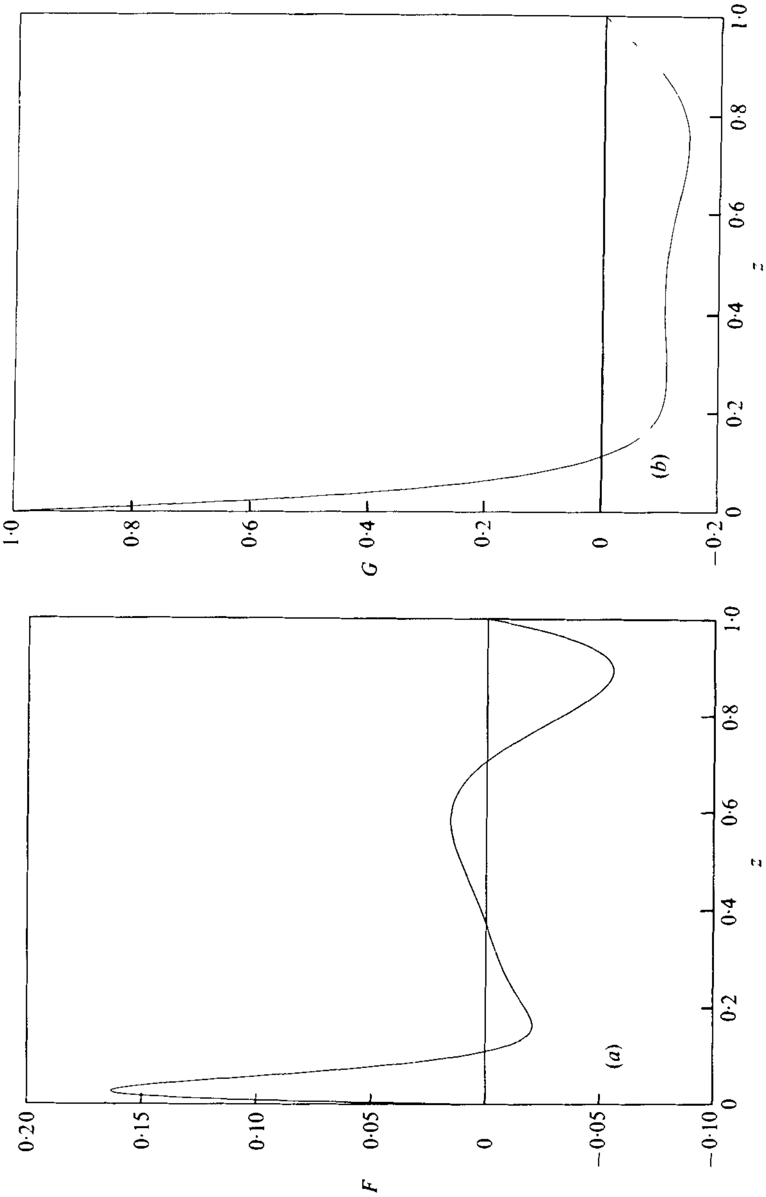
which is strictly empirical, is accurate to better than 4% for $0 \leqslant a \leqslant 2$.

Traditionally, the solutions to two single-disk flow problems have been matched to yield a solution to the two-disk problem. Batchelor (1951) considered the flow between a rotating and a stationary disk (R large, $a = 0$) and proposed that near either disk the solution behaves like a suitably scaled version of the solution to the appropriate problem for a single rotating or stationary disk when the fluid is rotating at

ζ_B	F	G	H
0·0	0·0000	0·0000	0·0000
0·5	−0·3486	0·3834	−0·1944
1·0	−0·4787	0·7354	−0·6241
1·5	−0·4496	1·0133	−1·0987
2·0	−0·3287	1·1923	−1·4928
2·5	−0·1762	1·2721	−1·7458
3·0	−0·0361	1·2713	−1·8496
3·5	0·0663	1·2182	−1·8308
4·0	0·1226	1·1412	−1·7325
4·5	0·1371	1·0640	−1·5995
5·0	0·1210	1·0016	−1·4685
5·5	0·0878	0·9611	−1·3632
6·0	0·0499	0·9427	−1·2944
6·5	0·0162	0·9420	−1·2619
7·0	−0·0084	0·9530	−1·2589
7·5	−0·0223	0·9692	−1·2751
8·0	−0·0268	0·9857	−1·3003
8·5	−0·0243	0·9990	−1·3263
9·0	−0·0179	1·0077	−1·3476
9·5	−0·0102	1·0118	−1·3617
10·0	−0·0033	1·0121	−1·3683
10·5	0·0018	1·0099	−1·3689
11·0	0·0047	1·0065	−1·3654
11·5	0·0057	1·0030	−1·3600
12·0	0·0052	1·0002	−1·3546
12·5	0·0038	0·9984	−1·3500
13·0	0·0022	0·9975	−1·3469
13·5	0·0008	0·9974	−1·3455
14·0	−0·0003	0·9979	−1·3454
14·5	−0·0010	0·9986	−1·3460
15·0	−0·0012	0·9993	−1·3471
15·5	−0·0011	0·9999	−1·3483
16·0	−0·0008	1·0003	−1·3492
16·5	−0·0005	1·0005	−1·3499
17·0	−0·0002	1·0005	−1·3502
17·5	0·0001	1·0004	−1·3503
18·0	0·0002	1·0002	−1·3501
18·5	0·0002	1·0001	−1·3499
19·0	0·0002	1·0000	−1·3496
19·5	0·0002	0·9999	−1·3494
20·0	0·0001	0·9998	−1·3493
40·0	0·0000	1·0000	−1·3494
60·0	0·0000	1·0000	−1·3494
80·0	0·0000	1·0000	−1·3494
100·0	0·0000	1·0000	−1·3494

TABLE 3. The Bödewadt solution as a function of the dimensionless distance ζ_B from the stationary disk: $R = 10^4$ and $a = 1{\cdot}3494$. The values of the derivatives F_{ζ_B} and G_{ζ_B} at $\zeta_B = 0$ are given in the text.

infinity. The rotation rate at infinity in both single-disk problems corresponds to that in the core region in the two-disk problem and must be determined as part of the matching process. It is found by requiring the axial inflow far from the single rotating disk to equal the axial outflow far from the single stationary disk. This



FIGURES 3(*a*) and (*b*). For legend see facing page.

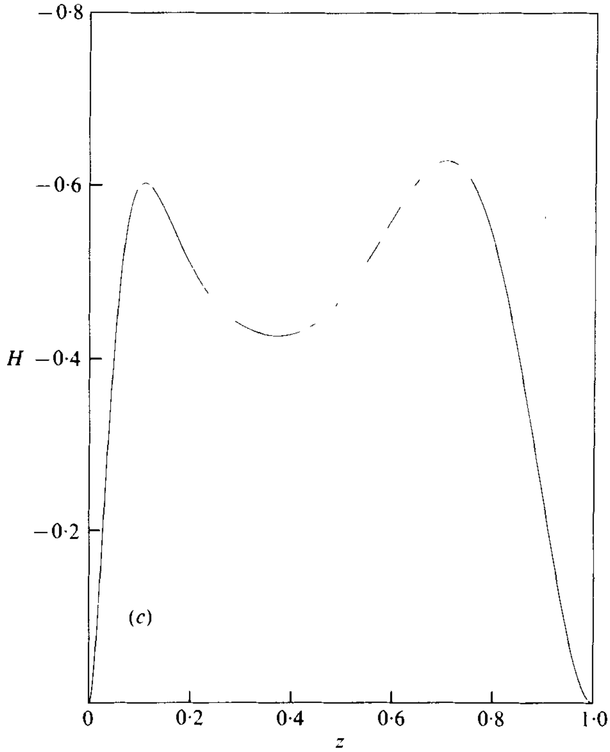


FIGURE 3. The equilibrium profiles of (a) F, (b) G and (c) H for $R = 10^3$, $a = 0$ and the rotational start-up scheme discussed in §5.

matching procedure was implemented by Lance & Rogers (1962) for $R = 441$, but with only moderate success since R was not large enough. The method was also discussed by H. P. Greenspan (1968). Reshotko & Rosenthal (1971) obtained more reliable results by using a matching procedure which invoked conservation of angular momentum as well as conservation of mass. For a different, but related, problem, McLeod & Parter (1974) showed rigorously that similar heuristic arguments about matching are correct.

Even though there is no need to obtain the two-disk flow solutions for $a > 0$ directly by such a matching procedure, the concept is still useful. For example, if R is sufficiently large, then for any $a \geqslant 0$ the flow near the stationary disk is a scaled version of Bödewadt flow (Bödewadt 1940). Let $F_B(\zeta_B)$, $G_B(\zeta_B)$ and $H(\zeta_B)$ represent the solution to Bödewadt's problem, where ζ_B measures distance from the stationary disk and $G_B \rightarrow 1$ far from the disk. Then if ω_c is the core angular velocity in a two-disk solution, the boundary-layer solution near the stationary disk should scale as follows (Reshotko & Rosenthal 1971):

$$F = \omega_c^{\frac{1}{2}} F_B, \quad G = \omega_c G_B, \quad H = \omega_c^{\frac{1}{2}} H_B, \quad \zeta = \omega_c^{-\frac{1}{2}} \zeta_B. \tag{4.2}$$

Our computational results do scale in such a manner. Hence knowledge of the matching procedure helps to explain why the boundary layer near the stationary disk becomes thinner and the oscillatory nature of the flow becomes more pronounced as the suction is increased.

That ω_c increases with a is also understandable: increased suction results in an increased axial inflow to the boundary layer near the rotating disk. Since this boundary-layer solution is to be matched to a scaled Bödewadt solution, it follows from (4.2) that ω_c must increase.

We used (4.2) and our computations for $R = 10^4$ and $a = 0$ to determine that $\omega_c = 1$ when $a = 1{\cdot}3494$. A subsequent computation with this value of a and with $R = 10^4$ yielded the Bödewadt solution, which is tabulated in table 3 as a function of ζ_B. In our notation $\zeta_B = R^{\frac{1}{2}}(1-z)$. Hence the dimensionless variable $\zeta_B = z^*(\omega/\nu)^{\frac{1}{2}}$, where z^* represents the dimensional distance from the stationary disk. The solution is accurate to at least 0·1%. The results extend those found in Schlichting (1968). We also find that $F_{\zeta_B}(0) = 0{\cdot}942$ and $G_{\zeta_B}(0) = 0{\cdot}773$.

5. Other solutions

A variety of numerical solutions have been obtained for the steady-state flow between a rotating and a stationary disk. Some (Mellor *et al.* 1968; Roberts & Shipman 1976) are multiple-cell solutions, for which the axial velocity vanishes on one or more planes parallel to the disks. However, Parter (1977, private communication) has recently shown analytically that if R is sufficiently large at least some of these multiple-cell solutions are not possible.

Mellor *et al.* also found other single-cell solutions. (Their results, which are for $a = 0$, presumably can be extended to the case $a > 0$.) One wonders whether a solution other than that exhibited in the preceding section can be obtained as an equilibrium solution to the *time-dependent* equations of motion. Do different rotational start-up schemes lead to different solutions?

We experimented briefly with schemes in which both disks rotated. We first gradually counter-rotated the disk at $z = 1$ (which is ultimately to be at rest) up to some designated speed and held it there until the flow reached equilibrium in the manner discussed previously. We tried counter-rotation rates one-tenth and one-twentieth of that finally achieved by the disk at $z = 0$. Then, again using an R^4 ramping time, we returned the disk at $z = 1$ to rest and brought the disk at $z = 0$ up to its normalized speed of unity. Finally, in a manner similar to that discussed earlier, we computed for an additional $200R^4$ time units. The final equilibrium results were the same in both instances and are shown in figures 3(*a*), (*b*) and (*c*). The computations were performed with $R = 10^3$ and with a specified error tolerance of 1·0%. Notice that all of the fluid except that in a boundary layer near the rotating disk has an angular velocity *opposite* in direction to that of the disk. This solution appears to be in reasonable agreement with that shown by Mellor *et al.* (after suitable renormalization) in figure 5 of their paper (for $R = -958$, in their notation).

Our work is by no means exhaustive, but it does demonstrate that more than one equilibrium solution to the time-dependent equations of motion can be found. Whether rotational schemes exist which lead to other solutions remains an open question.

A question which should be given serious consideration is whether (and, if so, under what circumstances) these idealized flows represent an adequate approximation to the real flows in a finite disk configuration. Even though the solutions may be correct from an analytical point of view, it appears to us that some, such as the counter-rotating solution discussed above, may be physically unacceptable.

We wish to thank S. V. Parter for several enlightening discussions and M. A. Gatto for producing the computer-generated figures. We also are greatly indebted to the reviewer for his or her careful reading of the paper and valuable suggestions.

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